

Non-cocommutative C^* -bialgebra defined as the direct sum of free group C^* -algebras

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Abstract

Let \mathbb{F}_n be the free group of rank n and let $\bigoplus C^*(\mathbb{F}_n)$ denote the direct sum of full group C^* -algebras $C^*(\mathbb{F}_n)$ of \mathbb{F}_n ($1 \leq n < \infty$). We introduce a new comultiplication Δ_φ on $\bigoplus C^*(\mathbb{F}_n)$ such that $(\bigoplus C^*(\mathbb{F}_n), \Delta_\varphi)$ is a non-cocommutative C^* -bialgebra, and $C^*(\mathbb{F}_\infty)$ is a comodule- C^* -algebra of $(\bigoplus C^*(\mathbb{F}_n), \Delta_\varphi)$. With respect to Δ_φ , tensor product formulas of several representations of \mathbb{F}_n 's are computed. From these results, a similarity between $C^*(\mathbb{F}_n)$'s and Cuntz algebras are discussed.

Mathematics Subject Classifications (2010).46K10, 16T10.

Key words. free group C^* -algebra; C^* -bialgebra; comodule- C^* -algebra; tensor product.

1 Introduction

A C^* -bialgebra is a generalization of bialgebra in the theory of C^* -algebras, which was introduced in C^* -algebraic framework for quantum groups [16, 17]. For example, if G is a locally compact group, then the full group C^* -algebra $C^*(G)$ of G is a cocommutative C^* -bialgebra with respect to the standard (diagonal) comultiplication. In this paper, we construct a new comultiplication Δ_φ on the direct sum

$$\bigoplus C^*(\mathbb{F}_n) = C^*(\mathbb{F}_1) \oplus C^*(\mathbb{F}_2) \oplus C^*(\mathbb{F}_3) \oplus \cdots \quad (1.1)$$

for all finite-rank free groups $\{\mathbb{F}_n : 1 \leq n < \infty\}$ such that $(\bigoplus C^*(\mathbb{F}_n), \Delta_\varphi)$ is a non-cocommutative C^* -bialgebra without antipode.

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In this section, we show our motivation, definitions and the main theorem.

1.1 Motivation

We have studied a new kind of C^* -bialgebras which are defined as direct sums of well-known C^* -algebras, for example, Cuntz algebras, UHF algebras, matrix algebras [13] and Cuntz-Krieger algebras [14]. They are non-commutative and non-cocommutative, and there never exist antipodes on them. Such bialgebra structures do not appear before one takes direct sums. With respect to their comultiplications, new tensor products among representations of these C^* -algebras were obtained [12, 15]. In [13], we gave a general method to construct a C^* -bialgebra from a given system of C^* -algebras and special $*$ -homomorphisms among them. The essential part of this construction is how to construct such $*$ -homomorphisms for each concrete example. One of our interests is to construct new examples of C^* -bialgebra from various C^* -algebras.

On the other hand, group C^* -algebras are important examples of C^* -algebras [3, 5, 18]. Furthermore, quantum groups in the C^* -algebra approach are founded on the study of group C^* -algebras [16, 17].

Hence we consider to construct a new C^* -bialgebra associated with group C^* -algebras by using a new comultiplication instead of their standard comultiplications. In this paper, we choose free group C^* -algebras for this purpose, and try to construct a new comultiplication on them according to our method [13].

1.2 C^* -bialgebra

In this subsection, we review terminology about C^* -bialgebra according to [8, 16, 17]. For two C^* -algebras A and B , we write $\text{Hom}(A, B)$ as the set of all $*$ -homomorphisms from A to B . We assume that every tensor product \otimes as below means the minimal C^* -tensor product.

Definition 1.1 *A pair (A, Δ) is a C^* -bialgebra if A is a C^* -algebra and $\Delta \in \text{Hom}(A, M(A \otimes A))$, where $M(A \otimes A)$ denotes the multiplier algebra of $A \otimes A$, such that the linear span of $\{\Delta(a)(b \otimes c) : a, b, c \in A\}$ is norm dense in $A \otimes A$ and the following holds:*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \quad (1.2)$$

We call Δ the comultiplication of A .

We say that a C^* -bialgebra (A, Δ) is *strictly proper* if $\Delta(a) \in A \otimes A$ for any $a \in A$; (A, Δ) is *unital* if A is unital and Δ is unital; (A, Δ) is *counital* if there exists $\varepsilon \in \text{Hom}(A, \mathbb{C})$ such that

$$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta. \quad (1.3)$$

We call ε the *counit* of A and write (A, Δ, ε) as the counital C^* -bialgebra (A, Δ) with the counit ε . Remark that Definition 1.1 does not mean $\Delta(A) \subset A \otimes A$. If A is unital, then (A, Δ) is strictly proper. A *bialgebra* in the purely algebraic theory [1, 11] means a unital counital strictly proper bialgebra with the unital counit with respect to the algebraic tensor product, which does not need to have an involution. Hence a C^* -bialgebra is not a bialgebra in general.

According to [13], we recall several notions of C^* -bialgebra.

Definition 1.2 (i) For two C^* -bialgebras (A_1, Δ_1) and (A_2, Δ_2) , f is a C^* -bialgebra morphism from (A_1, Δ_1) to (A_2, Δ_2) if f is a non-degenerate $*$ -homomorphism from A_1 to $M(A_2)$ such that $(f \otimes f) \circ \Delta_1 = \Delta_2 \circ f$. In addition, if $f(A_1) \subset A_2$, then f is called *strictly proper*.

(ii) A map f is a C^* -bialgebra endomorphism of a C^* -bialgebra (A, Δ) if f is a C^* -bialgebra morphism from A to A . In addition, if $f(A) \subset A$ and f is bijective, then f is called a C^* -bialgebra automorphism of (A, Δ) .

(iii) A pair (B, Γ) is a right comodule- C^* -algebra of a C^* -bialgebra (A, Δ) if B is a C^* -algebra and Γ is a non-degenerate $*$ -homomorphism from B to $M(B \otimes A)$ such that the following holds:

$$(\Gamma \otimes id) \circ \Gamma = (id \otimes \Delta) \circ \Gamma \quad (1.4)$$

where both $\Gamma \otimes id$ and $id \otimes \Delta$ are extended to unital $*$ -homomorphisms from $M(B \otimes A)$ to $M(B \otimes A \otimes A)$. The map Γ is called the *right coaction* of A on B .

(iv) A proper C^* -bialgebra (A, Δ) satisfies the cancellation law if $\Delta(A)(I \otimes A)$ and $\Delta(A)(A \otimes I)$ are dense in $A \otimes A$ where $\Delta(A)(I \otimes A)$ and $\Delta(A)(A \otimes I)$ denote the linear spans of sets $\{\Delta(a)(I \otimes b) : a, b \in A\}$ and $\{\Delta(a)(b \otimes I) : a, b \in A\}$, respectively.

Let $(B, m, \eta, \Delta, \varepsilon)$ be a bialgebra in the purely algebraic theory, where m is a multiplication and η is a unit of the algebra B . An endomorphism S of B is called an *antipode* for $(B, m, \eta, \Delta, \varepsilon)$ if S satisfies $m \circ (id \otimes S) \circ \Delta = \eta \circ \varepsilon = m \circ (S \otimes id) \circ \Delta$ [1, 11].

1.3 Free group algebras and homomorphisms among them

In this subsection, we briefly review free group C^* -algebras [3, 5], and introduce new homomorphisms among them in order to define a comultiplication.

For $n = \infty, 1, 2, 3, \dots$, let \mathbb{F}_n denote the free group of rank n where we use the symbol “ ∞ ” as the countable infinity for convenience in this paper. Let (\mathcal{K}_n, η_n) denote a direct sum of all irreducible representations (up to unitary equivalence) of the Banach algebra $\ell^1(\mathbb{F}_n)$. Let $C^*(\mathbb{F}_n)$ denote the *full group C^* -algebra* of \mathbb{F}_n , which is defined as the C^* -algebra generated by the image of $\ell^1(\mathbb{F}_n)$ by η_n . Remark that $C^*(\mathbb{F}_1)$ is $*$ -isomorphic to the C^* -algebra $C(\mathbb{T})$ of all complex-valued continuous functions on the torus \mathbb{T} . With respect to the natural identification of the group algebra $\mathbb{C}\mathbb{F}_n$ over the coefficient field \mathbb{C} with a subalgebra of $C^*(\mathbb{F}_n)$, $\mathbb{C}\mathbb{F}_n$ is dense in $C^*(\mathbb{F}_n)$. For $n = \infty, 1, 2, 3, \dots$, let $\{g_i^{(n)}\}$ be the free generators of \mathbb{F}_n . We also identify $g_i^{(n)}$ with the unitary $\eta_n(g_i^{(n)})$ in $C^*(\mathbb{F}_n)$.

We introduce $*$ -homomorphisms among $C^*(\mathbb{F}_n)$ ’s as follows.

Lemma 1.3 (i) *For $1 \leq n, m < \infty$, define the map $\varphi_{n,m}$ from $C^*(\mathbb{F}_{nm})$ to the minimal tensor product $C^*(\mathbb{F}_n) \otimes C^*(\mathbb{F}_m)$ by*

$$\varphi_{n,m}(g_{m(i-1)+j}^{(n)}) \equiv g_i^{(n)} \otimes g_j^{(m)} \quad (i = 1, \dots, n, j = 1, \dots, m). \quad (1.5)$$

Then it is well-defined on the whole of $C^(\mathbb{F}_{nm})$ as a unital $*$ -homomorphism.*

(ii) *For $1 \leq n < \infty$, define the map $\varphi_{\infty,n}$ from $C^*(\mathbb{F}_\infty)$ to the minimal tensor product $C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_n)$ by*

$$\varphi_{\infty,n}(g_{n(i-1)+j}^{(\infty)}) \equiv g_i^{(\infty)} \otimes g_j^{(n)} \quad (i \geq 1, j = 1, \dots, n). \quad (1.6)$$

Then it is well-defined on the whole of $C^(\mathbb{F}_\infty)$ as a unital $*$ -homomorphism.*

(iii) *If $n, m \geq 2$, then $\varphi_{n,m}$ is not injective.*

(iv) *Let $C_r^*(\mathbb{F}_n)$ denote the reduced group C^* -algebra of \mathbb{F}_n , which is defined as the C^* -algebra generated by the image of the left regular representation of \mathbb{F}_n . Then the map $\varphi_{n,m}$ in (1.5) can not be extended as a $*$ -homomorphism from $C_r^*(\mathbb{F}_{nm})$ to $C_r^*(\mathbb{F}_n) \otimes C_r^*(\mathbb{F}_m)$.*

Especially, $\varphi_{1,1}$ equals the standard comultiplication of $C^*(\mathbb{F}_1)$. The proof of Lemma 1.3 will be given in § 2.2.

1.4 Main theorem

In this subsection, we show our main theorem. Let $C^*(\mathbb{F}_n)$, $\{g_i^{(n)}\}_{i=1}^n$, $\mathbb{C}\mathbb{F}_n$, $\{\varphi_{n,m}\}_{n,m \geq 1}$ and $\{\varphi_{\infty,n}\}_{n \geq 1}$ be as in § 1.3.

Theorem 1.4 *Define the C^* -algebra \mathcal{A} as the direct sum*

$$\mathcal{A} \equiv \bigoplus_{1 \leq n < \infty} C^*(\mathbb{F}_n) \quad (1.7)$$

and define $\Delta_\varphi \in \text{Hom}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ and $\varepsilon \in \text{Hom}(\mathcal{A}, \mathbb{C})$ by

$$\Delta_\varphi(x) \equiv \sum_{m,l; ml=n} \varphi_{m,l}(x) \quad \text{when } x \in C^*(\mathbb{F}_n), \quad (1.8)$$

$$\varepsilon(x) \equiv \begin{cases} \varepsilon_1(x) & \text{when } x \in C^*(\mathbb{F}_1), \\ 0 & \text{when } x \in \bigoplus_{n \geq 2} C^*(\mathbb{F}_n), \end{cases} \quad (1.9)$$

where $\varepsilon_1 : C^*(\mathbb{F}_1) \rightarrow \mathbb{C}$ is defined as $\varepsilon_1|_{\mathbb{F}_1} = 1$. Then the following holds:

- (i) The C^* -algebra \mathcal{A} is a strictly proper counital C^* -bialgebra with the comultiplication Δ_φ and the counit ε .
- (ii) The C^* -bialgebra $(\mathcal{A}, \Delta_\varphi)$ satisfies the cancellation law.
- (iii) By the smallest unitization, $(\mathcal{A}, \Delta_\varphi, \varepsilon)$ can be extended to the unital counital C^* -bialgebra $(\tilde{\mathcal{A}}, \hat{\Delta}_\varphi, \tilde{\varepsilon})$.
- (iv) There never exists any antipode for any dense unital counital subbialgebra of $(\tilde{\mathcal{A}}, \hat{\Delta}_\varphi, \tilde{\varepsilon})$ in (iii).
- (v) Define the algebraic direct sum

$$\mathbb{C}\mathbb{F}_* \equiv \bigoplus_{alg} \{\mathbb{C}\mathbb{F}_n : 1 \leq n < \infty\}. \quad (1.10)$$

Then $\Delta_\varphi(\mathbb{C}\mathbb{F}_*) \subset \mathbb{C}\mathbb{F}_* \odot \mathbb{C}\mathbb{F}_*$ where \odot means the algebraic tensor product, and $\mathbb{C}\mathbb{F}_*$ is identified with a $*$ -subalgebra of \mathcal{A} with respect to the canonical embedding.

- (vi) Define $\Gamma_\varphi \in \text{Hom}(C^*(\mathbb{F}_\infty), M(C^*(\mathbb{F}_\infty) \otimes \mathcal{A}))$ by

$$\Gamma_\varphi(x) \equiv \prod_{1 \leq n < \infty} \varphi_{\infty,n}(x) \quad (x \in C^*(\mathbb{F}_\infty)), \quad (1.11)$$

where we identify the multiplier $M(C^*(\mathbb{F}_\infty) \otimes \mathcal{A})$ with the direct product $\prod_{n \geq 1} C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_n)$. Then $C^*(\mathbb{F}_\infty)$ is a right comodule- C^* -algebra of $(\mathcal{A}, \Delta_\varphi)$ with respect to the coaction Γ_φ .

Remark 1.5 (i) The R.H.S. in (1.8) is always a finite sum when $x \in C^*(\mathbb{F}_n)$.

(ii) The C^* -bialgebra $(\mathcal{A}, \Delta_\varphi)$ is non-cocommutative. In fact, the following holds:

$$\Delta_\varphi(g_2^{(6)}) = g_1^{(1)} \otimes g_2^{(6)} + g_1^{(2)} \otimes g_2^{(3)} + g_1^{(3)} \otimes g_2^{(2)} + g_2^{(6)} \otimes g_1^{(1)}. \quad (1.12)$$

(iii) Let δ_n denote the standard (cocommutative) comultiplication of $C^*(\mathbb{F}_n)$. Then we see that

$$(\delta_n \otimes \delta_m) \circ \varphi_{n,m} = (id_n \otimes \tau_{n,m} \otimes id_m) \circ (\varphi_{n,m} \otimes \varphi_{n,m}) \circ \delta_{nm} \quad (1.13)$$

for $n, m \geq 1$ where $\tau_{n,m}$ denotes the transposition of the tensor $C^*(\mathbb{F}_n) \otimes C^*(\mathbb{F}_m)$. The standard antipode of $C^*(\mathbb{F}_n)$ is not that of $(\mathcal{A}, \Delta_\varphi, \varepsilon)$.

(iv) In (1.7), every free group C^* -algebras $C^*(\mathbb{F}_n)$ ($1 \leq n < \infty$) appear at once. This is an essentially new structure of the class of free group C^* -algebras. On the other hand, $C^*(\mathbb{F}_\infty)$ appears as a comodule- C^* -algebra of $(\mathcal{A}, \Delta_\varphi)$. This shows a certain naturality of this bialgebra structure.

(v) As a C^* -algebra, $C^*(\mathbb{F}_n)$ and the Cuntz algebra \mathcal{O}_n [4] are quite different in the simplicity, amenability, projections, trace and K -groups. However, we find a similarity between two *classes* $C^*(\mathbb{F}_n)$'s and \mathcal{O}_n 's. The construction of $(\mathcal{A}, \Delta_\varphi, \varepsilon)$ is almost same as the C^* -bialgebra [13]:

$$\bigoplus \mathcal{O}_n = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3 \oplus \cdots. \quad (1.14)$$

In both cases, the essential part is given as the set of $*$ -homomorphisms defined by using canonical generators. They are perfectly same by exchanging their generators. As a comodule- C^* -algebra, $C^*(\mathbb{F}_\infty)$ and \mathcal{O}_∞ appears in each case. A different point is that the map $\varphi_{n,m}$ in (1.5) is not injective, but the corresponding map in (1.1) of [13] is injective. This unexpected similarity between two classes $C^*(\mathbb{F}_n)$'s and \mathcal{O}_n 's is interesting. In Proposition 3.1, this similarity will be strengthened from tensor product formulas of their representations, that is, for some small classes of their irreducible representations, their tensor product formulas coincide perfectly.

Problem 1.6 (i) From Lemma 1.3(iv), our construction does not work for $C_r^*(\mathbb{F}_n)$'s. By another method, construct a non-cocommutative C^* -bialgebra from $C_r^*(\mathbb{F}_n)$'s.

- (ii) For other class of groups, construct a C^* -bialgebra from group C^* -algebras without use of their standard comultiplications. For group algebras of symmetric groups and braid groups, our construction still do not work with respect to their standard generators.
- (iii) Set up an apparent reason about the similarity between $C^*(\mathbb{F}_n)$'s and \mathcal{O}_n 's in Remark 1.5(v). As an example of correspondence between representations of different algebraic systems, Weyl's unitary trick (or unitarian trick) is known [20]. Or more specifically, the class of free semigroup algebras [6, 7] may give the solution as a unification theory of $C^*(\mathbb{F}_n)$'s and \mathcal{O}_n 's.

In § 2, we prove Theorem 1.4. In § 3, we show tensor product formulas of representations of \mathbb{F}_n 's with respect to Δ_φ , and show some C^* -bialgebra automorphisms.

2 Proofs of theorems

In this section, we prove Lemma 1.3 and Theorem 1.4.

2.1 C^* -weakly coassociative system

According to § 3 in [13], we recall a general method to construct a C^* -bialgebra from a set of C^* -algebras and $*$ -homomorphisms among them. A *monoid* is a set M equipped with a binary associative operation $M \times M \ni (a, b) \mapsto ab \in M$, and a unit with respect to the operation. For example, $\mathbb{N} = \{1, 2, 3, \dots\}$ is an abelian monoid with respect to the multiplication. In order to show Theorem 1.4, we give a new definition of C^* -weakly coassociative system which is a generalization of the older in Definition 3.1 of [13].

Definition 2.1 *Let M be a monoid with the unit e . A data $\{(A_a, \varphi_{a,b}) : a, b \in M\}$ is a C^* -weakly coassociative system ($= C^*$ -WCS) over M if A_a is a unital C^* -algebra for $a \in M$ and $\varphi_{a,b}$ is a unital $*$ -homomorphism from A_{ab} to $A_a \otimes A_b$ for $a, b \in M$ such that*

- (i) *for all $a, b, c \in M$, the following holds:*

$$(id_a \otimes \varphi_{b,c}) \circ \varphi_{a,bc} = (\varphi_{a,b} \otimes id_c) \circ \varphi_{ab,c} \quad (2.1)$$

where id_x denotes the identity map on A_x for $x = a, c$,

- (ii) *there exists a counit ε_e of A_e such that $(A_e, \varphi_{e,e}, \varepsilon_e)$ is a counital C^* -bialgebra,*

(iii) for each $a \in \mathbf{M}$, the following holds:

$$(\varepsilon_e \otimes id_a) \circ \varphi_{e,a} = id_a = (id_a \otimes \varepsilon_e) \circ \varphi_{a,e}. \quad (2.2)$$

The condition (2.2) is weaker than the older, “ $\varphi_{e,a}(x) = I_e \otimes x$ and $\varphi_{a,e}(x) = x \otimes I_e$ for $x \in A_a$ and $a \in \mathbf{M}$ ” ([13], Definition 3.1). In fact, the older definition satisfies (2.2). From the new definition, the same result holds as follows.

Theorem 2.2 ([13], Theorem 3.1). *Let $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$ be a C^* -WCS over a monoid \mathbf{M} . Assume that \mathbf{M} satisfies that*

$$\#\mathcal{N}_a < \infty \text{ for each } a \in \mathbf{M} \quad (2.3)$$

where $\mathcal{N}_a \equiv \{(b, c) \in \mathbf{M} \times \mathbf{M} : bc = a\}$. Define C^* -algebras

$$A_* \equiv \oplus\{A_a : a \in \mathbf{M}\}, \quad C_a \equiv \oplus\{A_b \otimes A_c : (b, c) \in \mathcal{N}_a\} \quad (a \in \mathbf{M}).$$

Define $\Delta_\varphi^{(a)} \in \text{Hom}(A_a, C_a)$, $\Delta_\varphi \in \text{Hom}(A_*, A_* \otimes A_*)$ and $\varepsilon \in \text{Hom}(A_*, \mathbb{C})$ by

$$\begin{aligned} \Delta_\varphi^{(a)}(x) &\equiv \sum_{(b,c) \in \mathcal{N}_a} \varphi_{b,c}(x) \quad (x \in A_a), \quad \Delta_\varphi \equiv \oplus\{\Delta_\varphi^{(a)} : a \in \mathbf{M}\}, \\ \varepsilon(x) &\equiv \begin{cases} 0 & \text{when } x \in \oplus\{A_a : a \in \mathbf{M} \setminus \{e\}\}, \\ \varepsilon_e(x) & \text{when } x \in A_e. \end{cases} \end{aligned} \quad (2.4)$$

Then $(A_*, \Delta_\varphi, \varepsilon)$ is a strictly proper counital C^* -bialgebra.

We call $(A_*, \Delta_\varphi, \varepsilon)$ in Theorem 2.2 by a (counital) C^* -bialgebra associated with $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$. By using (2.2), we prove that ε is a counit of A_* in Theorem 2.2. We will prove Theorem 2.2 in Appendix A by using Definition 2.1.

The following lemma holds independently of the generalization in Definition 2.1(iii).

Lemma 2.3 *For the following C^* -WCS $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$, we assume the condition (2.3).*

- (i) ([13], Lemma 2.2) *For a given strictly proper non-unital counital C^* -bialgebra (A, Δ, ε) , let $\tilde{A} \equiv A \oplus \mathbb{C}$ denote the smallest unitization of A . Then there exist a unique extension $(\hat{\Delta}, \hat{\varepsilon})$ of (Δ, ε) on \tilde{A} such that $(\tilde{A}, \hat{\Delta}, \hat{\varepsilon})$ is a strictly proper unital counital C^* -bialgebra.*

- (ii) ([13], Lemma 3.2) For a C^* -WCS $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$ over \mathbf{M} , let $(A_*, \Delta_\varphi, \varepsilon)$ be as in Theorem 2.2 and let $(\tilde{A}_*, \hat{\Delta}_\varphi, \tilde{\varepsilon})$ be the smallest unitization of $(A_*, \Delta_\varphi, \varepsilon)$ in (i). Assume that any element in \mathbf{M} has no left inverse except the unit e . Then the antipode for any dense unital counital subbialgebra of $(\tilde{A}_*, \hat{\Delta}_\varphi, \tilde{\varepsilon})$ never exists.
- (iii) ([13], Lemma 3.1) Let $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$ be a C^* -WCS over a monoid \mathbf{M} and let (A_*, Δ_φ) be as in Theorem 2.2 associated with $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$. Define

$$X_{a,b} \equiv \varphi_{a,b}(A_{ab})(A_a \otimes I_b), \quad Y_{a,b} \equiv \varphi_{a,b}(A_{ab})(I_a \otimes A_b) \quad (a, b \in \mathbf{M}) \quad (2.5)$$

where $\varphi_{a,b}(A_{ab})(A_a \otimes I_b)$ and $\varphi_{a,b}(A_{ab})(I_a \otimes A_b)$ mean the linear spans of $\{\varphi_{a,b}(x)(y \otimes I_b) : x \in A_{ab}, y \in A_a\}$ and $\{\varphi_{a,b}(x)(I_a \otimes y) : x \in A_{ab}, y \in A_b\}$, respectively. If both $X_{a,b}$ and $Y_{a,b}$ are dense in $A_a \otimes A_b$ for each $a, b \in \mathbf{M}$, then (A_*, Δ_φ) satisfies the cancellation law.

- (iv) ([13], Theorem 3.2) For a C^* -WCS $\{(A_a, \varphi_{a,b}) : a, b \in \mathbf{M}\}$ over a monoid \mathbf{M} , assume that B is a unital C^* -algebra and a set $\{\varphi_{B,a} : a \in \mathbf{M}\}$ of unital $*$ -homomorphisms such that $\varphi_{B,a} \in \text{Hom}(B, B \otimes A_a)$ for each $a \in \mathbf{M}$ and the following holds:

$$(\varphi_{B,a} \otimes \text{id}_b) \circ \varphi_{B,b} = (\text{id}_B \otimes \varphi_{a,b}) \circ \varphi_{B,ab} \quad (a, b \in \mathbf{M}). \quad (2.6)$$

Then B is a right comodule- C^* -algebra of the C^* -bialgebra (A_*, Δ_φ) with the unital coaction $\Gamma_\varphi \equiv \prod_{a \in \mathbf{M}} \varphi_{B,a}$.

2.2 Homomorphisms among free groups

In this subsection, we consider some homomorphisms among free groups, and prove Lemma 1.3.

Let $g_1^{(n)}, \dots, g_n^{(n)}$ be the free generators of \mathbb{F}_n . For $n, m \geq 1$, define the group homomorphism $\phi_{n,m}$ from \mathbb{F}_{nm} to $\mathbb{F}_n \times \mathbb{F}_m$ by

$$\phi_{n,m}(g_{m(i-1)+j}^{(nm)}) \equiv (g_i^{(n)}, g_j^{(m)}) \quad (i = 1, \dots, n, j = 1, \dots, m). \quad (2.7)$$

The map $\phi_{n,m}$ is well-defined by the universality of \mathbb{F}_{nm} .

We show a lemma about $\phi_{n,m}$ as follows.

Lemma 2.4 *For each $n \geq 1$, we write 1 as the unit of \mathbb{F}_n .*

- (i) *For any $x \in \mathbb{F}_n$, there exists $(y, z) \in \mathbb{F}_m \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z) = (x, y)$.*

- (ii) For any $y \in \mathbb{F}_m$, there exists $(x, z) \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z) = (x, y)$.
- (iii) For any $(x, y) \in \mathbb{F}_n \times \mathbb{F}_m$, there exists $(x', z) \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z)(x', 1) = (x, y)$.
- (iv) For any $(x, y) \in \mathbb{F}_n \times \mathbb{F}_m$, there exists $(y', z) \in \mathbb{F}_m \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z)(1, y') = (x, y)$.
- (v) When $n, m \geq 2$, $\phi_{n,m}$ is not injective.

Proof. (i) Let $a_1, \dots, a_n, b_1, \dots, b_m, c_1, \dots, c_{nm}$ be the free generators of $\mathbb{F}_n, \mathbb{F}_m, \mathbb{F}_{nm}$, respectively. Assume that $x \in \mathbb{F}_n$ is written as a reduced word

$$x = a_{i_1}^{\varepsilon_1} \cdots a_{i_l}^{\varepsilon_l} \quad (2.8)$$

where $\varepsilon_i = 1$ or -1 for $i = 1, \dots, l$. For example, define $(y, z) \in \mathbb{F}_m \times \mathbb{F}_{nm}$ by

$$y \equiv b_1^{\varepsilon_1} \cdots b_1^{\varepsilon_l}, \quad z \equiv c_{m(i_1-1)+1}^{\varepsilon_1} \cdots c_{m(i_l-1)+1}^{\varepsilon_l}, \quad (2.9)$$

where y belongs to the abelian subgroup generated by the single element b_1 , and it is not always a reduced word in \mathbb{F}_m . Then the statement holds for (y, z) in (2.9).

- (ii) As the proof of (i), this is proved.
- (iii) From (ii), we can find $(x'', z) \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z) = (x'', y)$. Define $x' \equiv (x'')^{-1}x$, then the statement holds.
- (iv) As the proof of (iii), this is proved from (i).
- (v) Let c_i be as in the proof of (i). Define $x(i, l; j, k) \in \mathbb{F}_{nm}$ by

$$x(i, l; j, k) \equiv c_{m(i-1)+j} c_{m(i-1)+k}^{-1} c_{m(l-1)+k} c_{m(l-1)+j}^{-1} \quad (2.10)$$

for $i, l \in \{1, \dots, n\}$, $k, j \in \{1, \dots, m\}$. Then $x(i, l; j, k) \neq 1$ when $k \neq j$, $i \neq l$, but $x(i, l; j, k) \in \text{Ker} \phi_{n,m}$ for any i, l, j, k . ■

In the proof of Lemma 2.4(v), if $n = m = 2$, then the reduced word $c_1 c_2^{-1} c_4 c_3^{-1}$ in \mathbb{F}_4 satisfies

$$\phi_{2,2}(c_1 c_2^{-1} c_4 c_3^{-1}) = (1, 1). \quad (2.11)$$

We see that $\varphi_{n,m}$ in (1.5) is an extension of $\phi_{n,m}$.

Proof of Lemma 1.3 (i) Let (\mathcal{K}_n, η_n) be as in § 1.3. Define the unitary representation $\varphi_{n,m}^0$ of \mathbb{F}_{nm} on $\mathcal{K}_n \otimes \mathcal{K}_m$ by

$$\varphi_{n,m}^0(g_{m(i-1)+j}^{(nm)}) \equiv \eta_n(g_i^{(n)}) \otimes \eta_m(g_j^{(m)}) \quad (i = 1, \dots, n, j = 1, \dots, m). \quad (2.12)$$

The representation $\varphi_{n,m}^0$ is well-defined by the universality of \mathbb{F}_{nm} . Since the image of $\varphi_{n,m}^0$ is included in $C^*(\mathbb{F}_n) \otimes C^*(\mathbb{F}_m)$, $\varphi_{n,m}^0$ is uniquely extended to $\varphi_{n,m}$ in (1.5) such that

$$\varphi_{n,m}(\eta_{nm}(x)) = \varphi_{n,m}^0(x) \quad (2.13)$$

for each $x \in \mathbb{F}_{nm}$ ([3], Proposition 2.5.2). Hence the statement holds.

(ii) In analogy with (i), the statement holds.

(iii) For $x(i, l; j, k)$ in (2.10), we see that $\varphi_{n,m}(x(i, l; j, k) - 1) = 0$ for each i, l, j, k . Hence the statement holds.

(iv) If such an extension $\tilde{\varphi}_{n,m}$ of $\varphi_{n,m}$ exists, then $\tilde{\varphi}_{n,m}$ must be injective because $C_r^*(\mathbb{F}_{nm})$ is simple when $nm \geq 2$, ([5], Corollary VII.7.5 and its proof). On the other hand, $\tilde{\varphi}_{n,m}$ never be injective from (iii). ■

Problem 2.5 For the map $\phi_{n,m}$ in (2.7), $G_{n,m} \equiv \text{Ker} \phi_{n,m}$ is also a free group from the Nielsen-Schreier theorem ([19], Theorem 11.44). Determine the rank of $G_{n,m}$.

2.3 Proof of Theorem 1.4

We prove Theorem 1.4 in this subsection. Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Remark that (2.3) holds for any element in the monoid \mathbb{N} .

(i) From Theorem 2.2, it is sufficient to show that $\{(C^*(\mathbb{F}_n), \varphi_{n,m}) : n, m \in \mathbb{N}\}$ is a C^* -WCS over the monoid \mathbb{N} . By the definition of $\varphi_{n,m}$ in (1.5), we can verify that

$$(\varphi_{n,m} \otimes id_l) \circ \varphi_{nm,l} = (id_n \otimes \varphi_{m,l}) \circ \varphi_{n,ml} \quad (n, m, l \in \mathbb{N}) \quad (2.14)$$

where id_a denotes the identity map on $C^*(\mathbb{F}_a)$ for $a = n, l$. Hence (2.1) is satisfied. On the other hand, since $\varepsilon_1|_{\mathbb{F}_1} = 1$,

$$\{(\varepsilon_1 \otimes id_n) \circ \varphi_{1,n}\}(g_j^{(n)}) = (\varepsilon_1 \otimes id_n)(g_1^{(1)} \otimes g_j^{(n)}) = \varepsilon_1(g_1^{(1)}) g_j^{(n)} = g_j^{(n)} \quad (2.15)$$

for each $j = 1, \dots, n$ and $n \in \mathbb{N}$. By the same token, we obtain $(id_n \otimes \varepsilon_1) \circ \varphi_{n,1} = id_n$. Hence (2.2) is verified. Therefore $\{(C^*(\mathbb{F}_n), \varphi_{n,m}) : n, m \in \mathbb{N}\}$ is a C^* -WCS over the monoid \mathbb{N} .

(ii) For $n, m \in \mathbb{N}$, define three subsets $P_{n,m}, Q_{n,m}, R_{n,m}$ of $C^*(\mathbb{F}_n) \otimes C^*(\mathbb{F}_m)$ by

$$P_{n,m} \equiv \{ \varphi_{n,m}(z)(x \otimes I_m) : x \in \mathbb{F}_n, z \in \mathbb{F}_{nm} \}, \quad (2.16)$$

$$Q_{n,m} \equiv \{ \varphi_{n,m}(z)(I_n \otimes y) : y \in \mathbb{F}_m, z \in \mathbb{F}_{nm} \}, \quad (2.17)$$

$$R_{n,m} \equiv \{ x \otimes y : x \in \mathbb{F}_n, y \in \mathbb{F}_m \}. \quad (2.18)$$

Then their linear spans are dense subspaces of $\varphi_{n,m}(C^*(\mathbb{F}_{nm}))(C^*(\mathbb{F}_n) \otimes I_m)$, $\varphi_{n,m}(C^*(\mathbb{F}_{nm}))(I_n \otimes C^*(\mathbb{F}_m))$ and $C^*(\mathbb{F}_n) \otimes C^*(\mathbb{F}_m)$, respectively. From Lemma 2.4(iii), it is sufficient to show that $R_{n,m} \subset P_{n,m}$ and $R_{n,m} \subset Q_{n,m}$.

We prove $R_{n,m} \subset P_{n,m}$ as follows: For $(x, y) \in \mathbb{F}_n \times \mathbb{F}_m$, there exists $(x', z) \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\phi_{n,m}(z)(x', 1) = (x, y)$ from Lemma 2.4(iii). By definitions of $\phi_{n,m}$ and $\varphi_{n,m}$, this implies $\varphi_{n,m}(z)(x' \otimes I_m) = x \otimes y$. Therefore $R_{n,m} \subset P_{n,m}$.

In a similar fashion, we obtain $R_{n,m} \subset Q_{n,m}$ from Lemma 2.4(iv). Hence the statement holds.

(iii) From (i) and Lemma 2.3(i), the statement holds.

(iv) Remark that the monoid \mathbb{N} has no left invertible element except the unit 1. From the proof of (i) and Lemma 2.3(ii), the statement holds.

(v) From the proof of (i) and the definition of $\varphi_{n,m}$ in (1.5), we see that $\varphi_{n,m}(\mathbb{C}\mathbb{F}_{nm}) \subset \mathbb{C}\mathbb{F}_n \odot \mathbb{C}\mathbb{F}_m$. This implies the statement.

(vi) By definition, we see that

$$(\varphi_{\infty,n} \otimes id_m) \circ \varphi_{\infty,m} = (id_\infty \otimes \varphi_{n,m}) \circ \varphi_{\infty,nm} \quad (n, m \in \mathbb{N}). \quad (2.19)$$

From Lemma 2.3(iv) for $\varphi_{C^*(\mathbb{F}_\infty),n} \equiv \varphi_{\infty,n}$, the statement holds. \blacksquare

3 Representations and automorphisms

In this section, we show tensor product formulas of unitary representations of \mathbb{F}_n 's with respect to the comultiplication Δ_φ in Theorem 1.4, and C^* -bialgebra automorphisms.

3.1 General facts about representations of \mathbb{F}_n and $C^*(\mathbb{F}_n)$

We identify \mathbb{F}_n with the unitary subgroup of $C^*(\mathbb{F}_n)$ with respect to the canonical embedding. Let $\text{Rep}_u \mathbb{F}_n$ denote the class of all unitary representations of \mathbb{F}_n . For $(\pi, \pi') \in \text{Rep}_u \mathbb{F}_n \times \text{Rep}_u \mathbb{F}_m$, define the new representation

$\pi \otimes_{\varphi} \pi' \in \text{Rep}_u \mathbb{F}_{nm}$ by

$$\pi \otimes_{\varphi} \pi' \equiv (\pi \otimes \pi') \circ \varphi_{n,m}, \quad (3.1)$$

where $\varphi_{n,m}$ is as in (1.5). Then we see that the new operation \otimes_{φ} is associative, and it is distributive with respect to the direct sum. Furthermore, \otimes_{φ} is well-defined on the unitary equivalence classes of representations. In a similar fashion, the new (associative) tensor product \otimes_{φ} of states on $C^*(\mathbb{F}_n)$'s is defined.

Let Φ_n be the canonical surjective $*$ -homomorphism from $C^*(\mathbb{F}_n)$ onto $C_r^*(\mathbb{F}_n)$. Then any representation π of $C_r^*(\mathbb{F}_n)$ is lifted to $C^*(\mathbb{F}_n)$ as

$$\tilde{\pi} \equiv \pi \circ \Phi_n. \quad (3.2)$$

Especially, if π is irreducible, then so is $\tilde{\pi}$. Likewise, a state (a pure state) on $C_r^*(\mathbb{F}_n)$ is lifted to that on $C^*(\mathbb{F}_n)$.

3.2 Tensor product of some irreducible representations

In this subsection, we show tensor product formulas of some irreducible representations of \mathbb{F}_n 's.

According to [2, 21], we review some irreducible unitary representations of \mathbb{F}_n . Let g_1, \dots, g_n be the free generators of \mathbb{F}_n . For $i \in \{1, \dots, n\}$, let $H_i^{(n)}$ denote the abelian subgroup of \mathbb{F}_n generated the single element g_i . Define the positive definite function $f_i^{(n)}$ on \mathbb{F}_n by

$$f_i^{(n)}(x) \equiv \begin{cases} 1 & x \in H_i^{(n)}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

Then the Gel'fand-Raikov representation of \mathbb{F}_n by $f_i^{(n)}$ is irreducible [9, 10, 21]. The GNS representation of $C^*(\mathbb{F}_n)$ by the unique pure state extension $\tilde{f}_i^{(n)}$ of $f_i^{(n)}$ is the extension of the Gel'fand-Raikov representation of \mathbb{F}_n by $f_i^{(n)}$. We identify $f_i^{(n)}$ with $\tilde{f}_i^{(n)}$. Hence we can define the operation \otimes_{φ} among $\{f_i^{(n)} : n \geq 1, i = 1, \dots, n\}$ and their GNS representations.

Proposition 3.1 *Let $n, m \geq 1$.*

- (i) *For $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$,*

$$f_i^{(n)} \otimes_{\varphi} f_j^{(m)} = f_{m(i-1)+j}^{(nm)}. \quad (3.4)$$

- (ii) Let $\mathbb{F}_n/H_i^{(n)}$ denote the left coset space, that is, $\mathbb{F}_n/H_i^{(n)} = \{xH_i^{(n)} : x \in \mathbb{F}_n\}$. Define the permutation representation $L_i^{(n)}$ of \mathbb{F}_n on $\ell^2(\mathbb{F}_n/H_i^{(n)})$ by the natural left action of \mathbb{F}_n on the standard basis of $\ell^2(\mathbb{F}_n/H_i^{(n)})$:

$$\mathbb{F}_n \overset{L_i^{(n)}}{\curvearrowright} \ell^2(\mathbb{F}_n/H_i^{(n)}). \quad (3.5)$$

Then the restriction of the GNS representation $(\mathcal{H}_i^{(n)}, \Pi_i^{(n)}, \Omega_i^{(n)})$ of $C^*(\mathbb{F}_n)$ by $f_i^{(n)}$ to \mathbb{F}_n is unitarily equivalent to $(\ell^2(\mathbb{F}_n/H_i^{(n)}), L_i^{(n)}, e_0)$ where $e_0 \in \ell^2(\mathbb{F}_n/H_i^{(n)})$ is the unit cyclic vector associated with the coset $H_i^{(n)} \in \mathbb{F}_n/H_i^{(n)}$.

- (iii) Let $\Pi_i^{(n)}$ denote $(\mathcal{H}_i^{(n)}, \Pi_i^{(n)}, \Omega_i^{(n)})$ in (ii) for the simplicity of description. If $i \neq j$, then $\Pi_i^{(n)} \not\cong \Pi_j^{(n)}$ where \cong means the unitary equivalence.
- (iv) For $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$,

$$\Pi_i^{(n)} \otimes_{\varphi} \Pi_j^{(m)} \cong \Pi_{m(i-1)+j}^{(nm)}. \quad (3.6)$$

Proof. (i) By definition, the statement holds.

(ii) Define the unitary W from $\mathcal{H}_i^{(n)}$ to $\ell^2(\mathbb{F}_n/H_i^{(n)})$ by $W\Pi_i^{(n)}(x)\Omega_i^{(n)} \equiv L_i^{(n)}(x)e_0$ for $x \in \mathbb{F}_n$. Then we can verify that W is well-defined as a unitary and $W\Pi_i^{(n)}(\cdot)W^* = L_i^{(n)}$.

(iii) Here we write $(\mathcal{H}_i^{(n)}, \Pi_i^{(n)}, \Omega_i^{(n)})$ as $(\mathcal{H}_i, \Pi_i, \Omega_i)$ for the simplicity of description. By (3.3), we see that

$$\Pi_i(g_i)\Omega_i = \Omega_i. \quad (3.7)$$

If there exists a unitary U from \mathcal{H}_i to \mathcal{H}_j such that $U\Pi_i(\cdot)U^* = \Pi_j$, then we can assume that both Ω_i and Ω_j in (3.7) belong to the same Hilbert space \mathcal{H} , and we write both $\Pi_i(x)$ and $\Pi_j(x)$ as x for $x \in \mathbb{F}_n$. Since Ω_i is a cyclic vector of \mathcal{H} , we can write $\Omega_j = \sum w_x x\Omega_i \in \mathcal{H}$ for $w_x \in \mathbb{C}$. From (ii), the sum is taken over representatives of cosets in $\mathbb{F}_n/H_i^{(n)}$. Then

$$\Omega_j = g_j^{-1}\Omega_j = \sum w_x g_j^{-1}x\Omega_i. \quad (3.8)$$

Since $i \neq j$, $g_j x \neq x$ for each x in this sum. From this, $w_x = w_{g_j x} = w_{g_j^2 x} = \dots$. Because $\|\Omega_j\| = 1 < \infty$, $w_x = 0$ for each x . Hence $\Omega_j = 0$. This is a contradiction. Therefore the statement holds.

(iv) Let $a_1, \dots, a_n, b_1, \dots, b_m, c_1, \dots, c_{nm}$ be the free generators of $\mathbb{F}_n, \mathbb{F}_m, \mathbb{F}_{nm}$, respectively. Here we write $(\mathcal{H}_i^{(n)}, \Pi_i^{(n)}, \Omega_i^{(n)})$ and $(\mathcal{H}_j^{(m)}, \Pi_j^{(m)}, \Omega_j^{(m)})$ as $(\mathcal{H}_i, \Pi_i, \Omega_i)$ and $(\mathcal{H}_j, \Pi_j, \Omega_j)$, respectively for the simplicity of description. We identify $\Pi_k(x)$ with x for $k = i, j$. From (3.3), we see that

$$a_i \Omega_i = \Omega_i, \quad b_j \Omega_j = \Omega_j. \quad (3.9)$$

This implies that

$$\varphi_{n,m}(c_{m(i-1)+j})(\Omega_i \otimes \Omega_j) = a_i \Omega_i \otimes b_j \Omega_j = \Omega_i \otimes \Omega_j. \quad (3.10)$$

From this, we see that

$$f_{m(i-1)+j}^{(nm)} = \langle \Omega_i \otimes \Omega_j | \varphi_{n,m}(\cdot) (\Omega_i \otimes \Omega_j) \rangle. \quad (3.11)$$

Define the cyclic subspace \mathcal{H}' of $\mathcal{H}_i \otimes \mathcal{H}_j$ by the action $\Pi_i \otimes_\varphi \Pi_j$ of \mathbb{F}_{nm} with the cyclic vector $\Omega_i \otimes \Omega_j$. Then $(\mathcal{H}', (\Pi_i \otimes_\varphi \Pi_j)|_{\mathcal{H}'}, \Omega_i \otimes \Omega_j)$ is unitarily equivalent to $(\mathcal{H}_{m(i-1)+j}^{(nm)}, \Pi_{m(i-1)+j}^{(nm)}, \Omega_{m(i-1)+j}^{(nm)})$. Hence if we show $\mathcal{H}' = \mathcal{H}_1 \otimes \mathcal{H}_2$, then the statement is proved. For this purpose, it is sufficient to show that, for any $(x, y) \in \mathbb{F}_n \times \mathbb{F}_m$, there exists $z \in \mathbb{F}_{nm}$ such that

$$x \Omega_i \otimes y \Omega_j = \varphi_{n,m}(z)(\Omega_i \otimes \Omega_j). \quad (3.12)$$

We show (3.12) as follows.

For any $(x, y) \in \mathbb{F}_n \times \mathbb{F}_m$, there exists $(x', z') \in \mathbb{F}_n \times \mathbb{F}_{nm}$ such that $\varphi_{n,m}(z')(x' \otimes I_m) = x \otimes y$ by Lemma 2.4(iii). Assume that x' is written as a reduced word

$$a_{i_1}^{\varepsilon_1} \cdots a_{i_l}^{\varepsilon_l} \quad (3.13)$$

for $\varepsilon_j = +1$ or -1 . Define $z'', z \in \mathbb{F}_{nm}$ by

$$z'' \equiv c_{m(i_1-1)+j}^{\varepsilon_1} \cdots c_{m(i_l-1)+j}^{\varepsilon_l}, \quad z \equiv z' z''. \quad (3.14)$$

Then

$$\begin{aligned} \varphi_{n,m}(z)(\Omega_i \otimes \Omega_j) &= \varphi_{n,m}(z') \varphi_{n,m}(z'')(\Omega_i \otimes \Omega_j) \\ &= \varphi_{n,m}(z')(x' \Omega_i \otimes (b_j^{\varepsilon_1} \cdots b_j^{\varepsilon_l}) \Omega_j) \\ &= \varphi_{n,m}(z')(x' \Omega_i \otimes \Omega_j) \quad (\text{from (3.9)}) \\ &= \varphi_{n,m}(z')(x' \otimes I_m)(\Omega_i \otimes \Omega_j) \\ &= x \Omega_i \otimes y \Omega_j. \end{aligned}$$

Hence (3.12) holds. From this, the statement is proved. ■

From Proposition 3.1(i), we obtain

$$f_1^{(2)} \otimes_{\varphi} f_2^{(3)} = f_2^{(6)}, \quad f_2^{(3)} \otimes_{\varphi} f_1^{(2)} = f_4^{(6)}. \quad (3.15)$$

Hence the operation \otimes_{φ} is not commutative. From Proposition 3.1(iii) and (iv), we see that the operation \otimes_{φ} among unitary equivalence classes of representations is not commutative.

The result in Proposition 3.1(iv) is perfectly same as the case of permutative representations of Cuntz algebras ([12], §4.1). This fact also implies a similarity between $C^*(\mathbb{F}_n)$'s and \mathcal{O}_n 's.

3.3 Tensor product of regular representations

In this subsection, we consider regular representations of \mathbb{F}_n 's.

Proposition 3.2 *For $n \geq 1$, let $\lambda^{(n)}$ denote the left regular representation of \mathbb{F}_n . Then the following holds:*

$$(i) \quad \lambda^{(n)} \otimes_{\varphi} \lambda^{(m)} \cong (\lambda^{(nm)})^{\oplus \infty} \quad (n, m \geq 1) \quad (3.16)$$

where \cong means the unitary equivalence of representations.

$$(ii) \quad \lambda^{(n)} \otimes_{\varphi} \lambda^{(m)} \sim \lambda^{(nm)} \quad (n, m \geq 1) \quad (3.17)$$

where \sim means the quasi-equivalence of representations.

Proof. (i) Let $\{g_i^{(n)}\}_{i=1}^n$ be the free generators of \mathbb{F}_n . Let $\phi_{n,m}$ be as in (2.7). For $x \in \mathbb{F}_n$, let $[x]$ denote the orbits in $\mathbb{F}_n \times \mathbb{F}_m$ with respect to the left multiplicative action of $\phi_{n,m}$ such that $(x, 1) \in [x]$. Then we see that $[x] = [x']$ if and only if $x = x'$ for $x, x' \in \mathbb{F}_n$.

Let $\{\xi_g^{(n)} : g \in \mathbb{F}_n\}$ denote the standard basis of the Hilbert space $\ell^2(\mathbb{F}_n)$. For $x \in \mathbb{F}_n$, define the closed subspace V_x of $\ell^2(\mathbb{F}_n) \otimes \ell^2(\mathbb{F}_m)$ as the closure of

$$\text{Lin}\langle \{(\lambda^{(n)} \otimes_{\varphi} \lambda^{(m)})(z)(\xi_x^{(n)} \otimes \xi_1^{(m)}) : z \in \mathbb{F}_{nm}\} \rangle. \quad (3.18)$$

Then we see that $V_x = V_{x'}$ if and only if $x = x'$. From Lemma 2.4(iii),

$$\text{Lin}\langle \{\xi_{x'}^{(n)} \otimes \xi_{y'}^{(m)} : (x', y') \in [x]\} \rangle \quad (3.19)$$

is dense in V_x . From this, V_x and $V_{x'}$ are orthogonal when $x \neq x'$ for $x, x' \in \mathbb{F}_n$. For $x \in \mathbb{F}_n$, define the unitary $U_{n,m,x}$ from $\ell^2(\mathbb{F}_{nm})$ to V_x by

$$U_{n,m,x} \xi_g^{(nm)} \equiv \xi_{g'_x}^{(n)} \otimes \xi_{g''}^{(m)} \quad \text{when } \phi_{n,m}(g) = (g', g'') \quad (3.20)$$

for $g \in \mathbb{F}_{nm}$. Then we can verify that

$$U_{n,m,x} \lambda_g^{(nm)} U_{n,m,x}^* = (\lambda^{(n)} \otimes_\varphi \lambda^{(m)})_g \quad (g \in \mathbb{F}_{nm}). \quad (3.21)$$

Since $\mathbb{F}_n \times \mathbb{F}_m = \coprod_{x \in \mathbb{F}_n} [x]$, we obtain the following decomposition

$$\ell^2(\mathbb{F}_n) \otimes \ell^2(\mathbb{F}_m) = \bigoplus_{x \in \mathbb{F}_n} V_x. \quad (3.22)$$

Because the subrepresentation $(V_x, (\lambda^{(n)} \otimes_\varphi \lambda^{(m)})|_{V_x})$ is unitarily equivalent to $(\ell^2(\mathbb{F}_{nm}), \lambda^{(nm)})$ for each $x \in \mathbb{F}_n$, we obtain (3.16).

(ii) From (i), the statement holds. \blacksquare

Proposition 3.2 shows that the set of quasi-equivalence classes of left regular representations of \mathbb{F}_n 's is an abelian semigroup with respect to the new tensor product \otimes_φ . This is also a new tensor product formula of Π_1 -factor representations. This also implies a naturality of \otimes_φ and Δ_φ .

3.4 Automorphisms

In this subsection, we show examples of some C^* -bialgebra automorphism of $(\mathcal{A}, \Delta_\varphi)$. For $t \in \mathbb{R}$, define $\alpha_t^{(n)} \in \text{Aut} C^*(\mathbb{F}_n)$ by

$$\alpha_t^{(n)}(g_i^{(n)}) \equiv e^{\sqrt{-1}t \log n} g_i^{(n)} \quad (i = 1, \dots, n). \quad (3.23)$$

Then $\alpha_t^{(*)} \equiv \bigoplus_{n \geq 1} \alpha_t^{(n)}$ is a C^* -bialgebra automorphism of $(\mathcal{A}, \Delta_\varphi)$ such that $\alpha_t^{(*)} \circ \alpha_s^{(*)} = \alpha_{t+s}^{(*)}$ for $s, t \in \mathbb{R}$.

Define $\beta^{(n)} \in \text{Aut} C^*(\mathbb{F}_n)$ by

$$\beta^{(n)}(g_i^{(n)}) \equiv g_{n-i+1}^{(n)} \quad (i = 1, \dots, n). \quad (3.24)$$

Then $\beta^{(*)} \equiv \bigoplus_{n \geq 1} \beta^{(n)}$ is a C^* -bialgebra automorphism of $(\mathcal{A}, \Delta_\varphi)$ such that $\beta^{(*)} \circ \beta^{(*)} = id$.

The automorphism $\beta^{(*)}$ commutes $\alpha_t^{(*)}$ for each t . Hence these give the action of the group $\mathbb{R} \times (\mathbb{Z}/2\mathbb{Z})$ on the C^* -bialgebra $(\mathcal{A}, \Delta_\varphi)$.

Appendix

A Proof of Theorem 2.2

We prove Theorem 2.2 here. By (2.3), $\Delta_\varphi^{(a)}$ is well-defined. Furthermore, C_a is unital and $\Delta_\varphi^{(a)}$ is unital for each a . Since $\mathbf{M} \times \mathbf{M} = \coprod_{a \in \mathbf{M}} \mathcal{N}_a$,

$$A_* \otimes A_* = \oplus \{A_f \otimes A_g : f, g \in \mathbf{M}\} = \oplus \{C_a : a \in \mathbf{M}\}. \quad (\text{A.1})$$

Since $\Delta_\varphi^{(a)}$ is unital for each a , Δ_φ is non-degenerate. From (2.1), the following holds for $x \in A_a$:

$$\begin{aligned} \{(\Delta_\varphi \otimes id) \circ \Delta_\varphi\}(x) &= \sum_{b,c,d \in \mathbf{M}, bcd=a} (\varphi_{b,c} \otimes id_d)(\varphi_{bc,d}(x)) \\ &= \sum_{b,c,d \in \mathbf{M}, bcd=a} (id_b \otimes \varphi_{c,d})(\varphi_{b,cd}(x)) \\ &= \{(id \otimes \Delta_\varphi) \circ \Delta_\varphi\}(x). \end{aligned} \quad (\text{A.2})$$

Hence $(\Delta_\varphi \otimes id) \circ \Delta_\varphi = (id \otimes \Delta_\varphi) \circ \Delta_\varphi$ on A_* . Therefore Δ_φ is a comultiplication of A_* . On the other hand, for $x \in A_a$, we see that

$$\begin{aligned} \{(\varepsilon \otimes id) \circ \Delta_\varphi\}(x) &= (\varepsilon \otimes id)(\Delta_\varphi^{(a)}(x)) \\ &= \sum_{(b,c) \in \mathcal{N}_a} (\varepsilon \otimes id)(\varphi_{b,c}(x)) \\ &= (\varepsilon_e \otimes id_a)(\varphi_{e,a}(x)) \\ &= x \quad (\text{from (2.2)}). \end{aligned} \quad (\text{A.3})$$

Hence $(\varepsilon \otimes id) \circ \Delta_\varphi = id$. In like wise, we see that $(id \otimes \varepsilon) \circ \Delta_\varphi = id$. Therefore ε is a counit of (A_*, Δ_φ) . In consequence, we see that $(A_*, \Delta_\varphi, \varepsilon)$ is a counital C^* -bialgebra. By definition, (A_*, Δ_φ) is strictly proper. \blacksquare

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